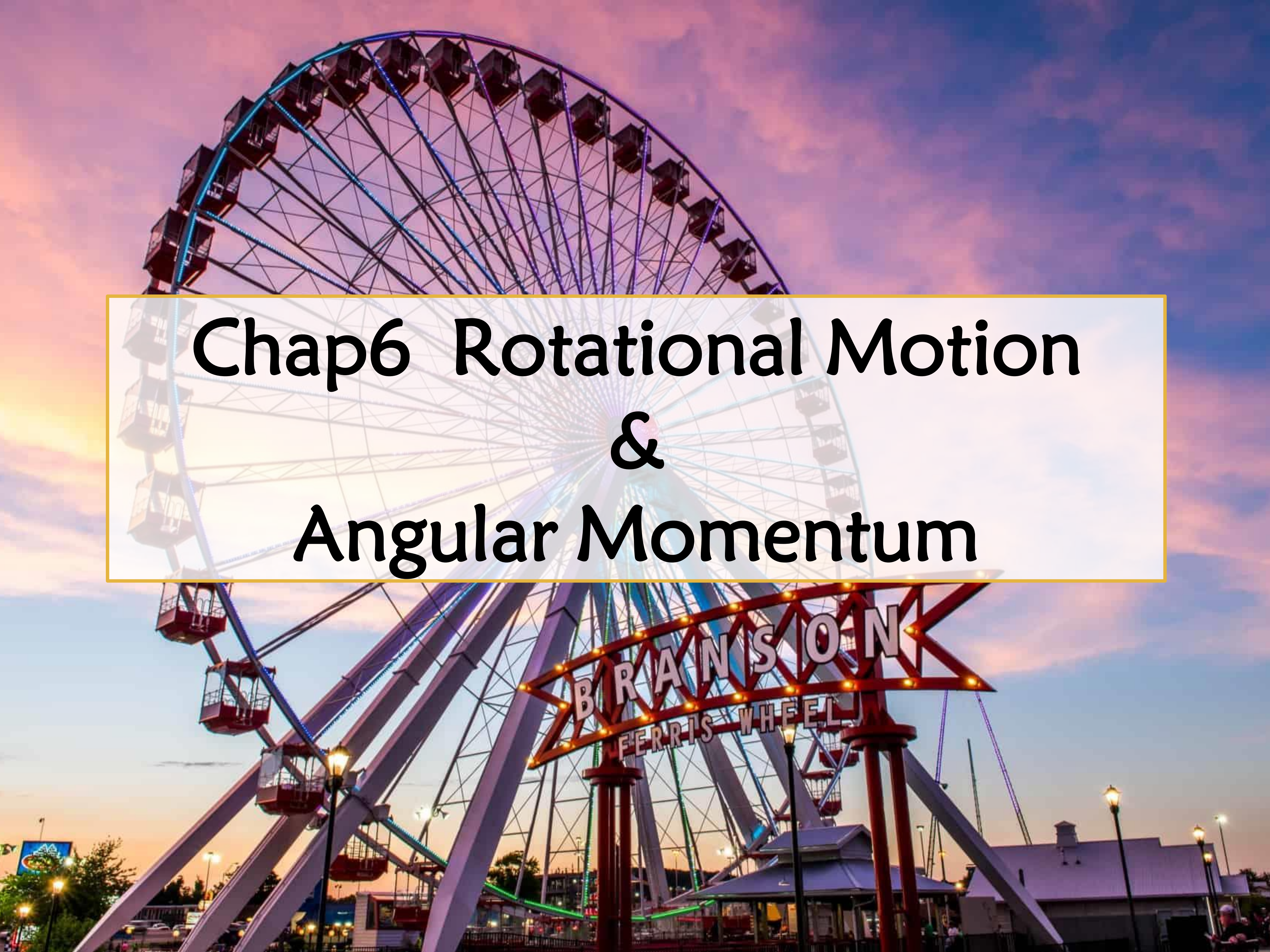
Chap6 Rotational Motion
&
Angular Momentum



Angular Quantities

vector

$$\theta = \frac{s}{r}, \quad \omega = \frac{v}{r}$$

vector

Angular velocity

$$\overline{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}, \quad \omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

Angular acceleration

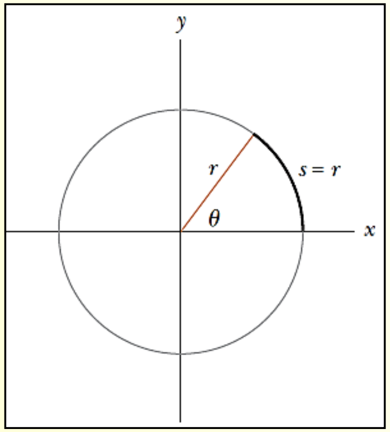
$$\overline{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}, \quad \alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

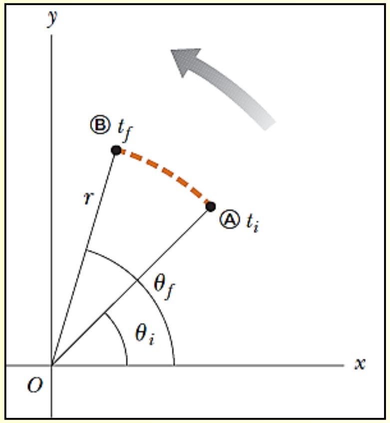
Centripetal (or radial) acceleration VS Tangential acceleration

$$a_R = v\omega = \frac{v^2}{r} = r\omega^2$$

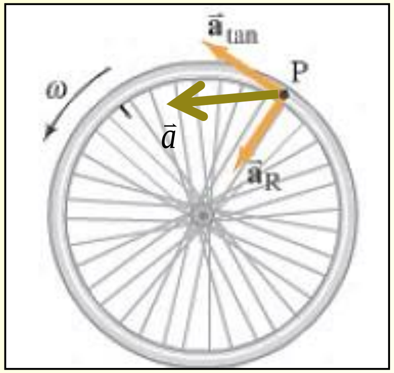
$$a_{tan} = \frac{\Delta v}{\Delta t} = r\frac{\Delta\omega}{\Delta t} = r\alpha$$

Linear	Type	Rotational	Relation
s	displacement	θ	$s = r\theta$
v	velocity	ω	$v = r\omega$
a_{tan}	acceleration	α	$a_{tan} = r\alpha$





$$\vec{a} = \vec{a}_R + \vec{a}_{tan}$$

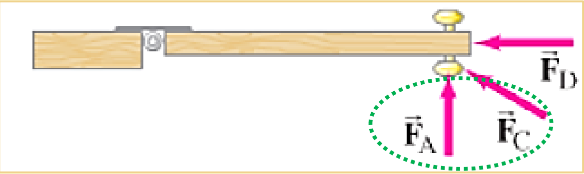


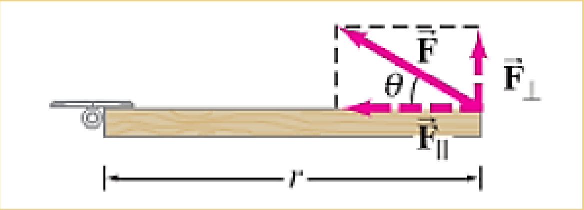
Constant Angular Acceleration

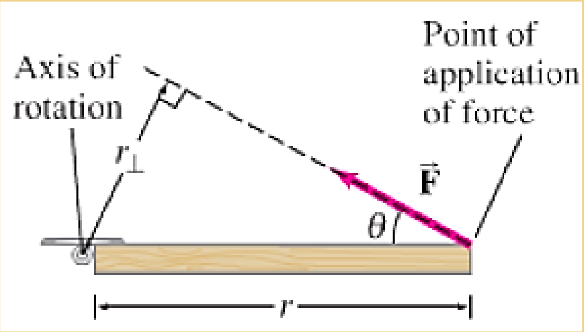
Kinematic Equations

Linear	Angular
$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$
$x = v_i t + \frac{1}{2} a t^2$	$\theta = \omega_i t + \frac{1}{2} \alpha t^2$
$v_f^2 - v_i^2 = 2ax$	$\omega_f^2 - \omega_i^2 = 2\alpha\theta$
$\bar{v} = \frac{v_i + v_f}{2}$	$\bar{\omega} = \frac{\omega_i + \omega_f}{2}$

Torque & Moment of Inertia







Torque

vector

$$\tau = rF_\perp = rF\sin\theta = r_\perp F$$

SI unit: m·N

$a \propto \Sigma F$ ⇒ $\alpha \propto \Sigma\tau$

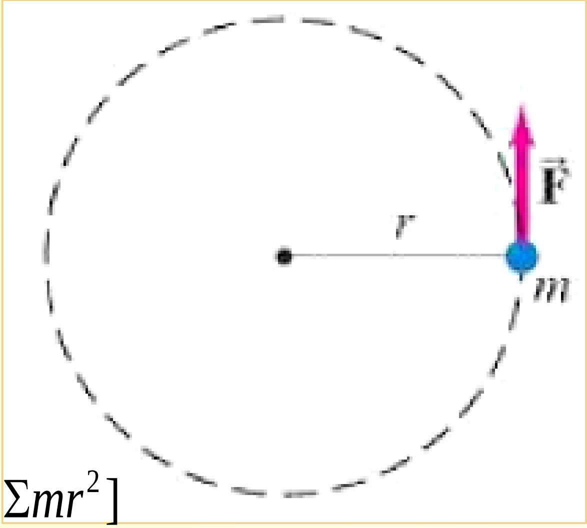
- As for single particle:

$$\left.\begin{aligned} F &= ma_{tan} \\ a_{tan} &= r\alpha \\ \tau &= rF \end{aligned}\right\} \Rightarrow \tau = mr^2\alpha = I\alpha \Rightarrow \tau = 0,\ \alpha = 0 \Rightarrow \omega = const.$$

***I*: moment of inertia**

Newton's Second Law for Rotation

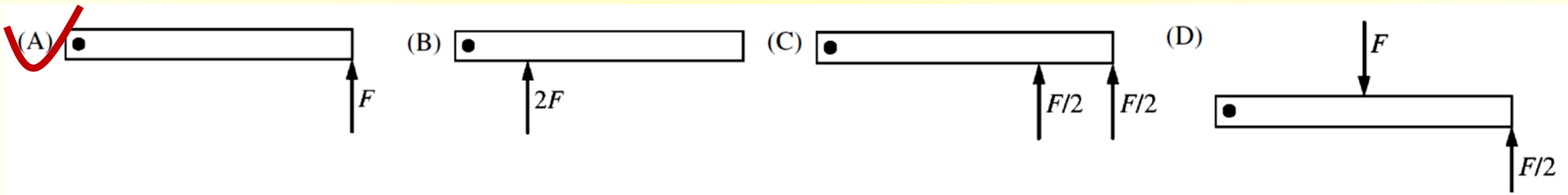
Newton's First Law for Rotation



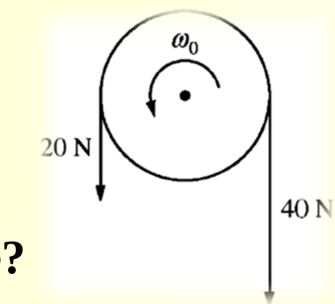
- For many particles (α = const.): $\Sigma\tau = (\Sigma mr^2)\alpha = I\alpha\ [I = \Sigma mr^2]$

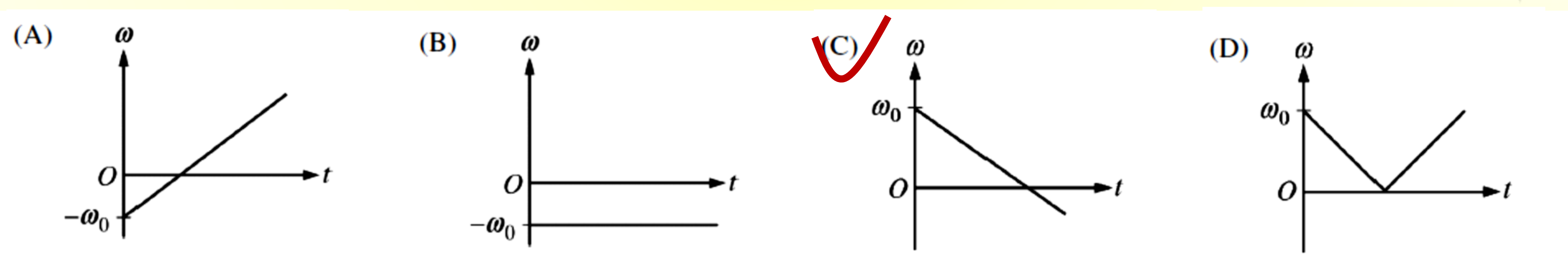
【Example】

A solid metal bar is at rest on a horizontal frictionless surface. It is free to rotate about a vertical axis at the left end. The figures below show forces of different magnitudes that are exerted on the bar at different locations. In which case does the bar's angular speed about the axis increase at the fastest rate?



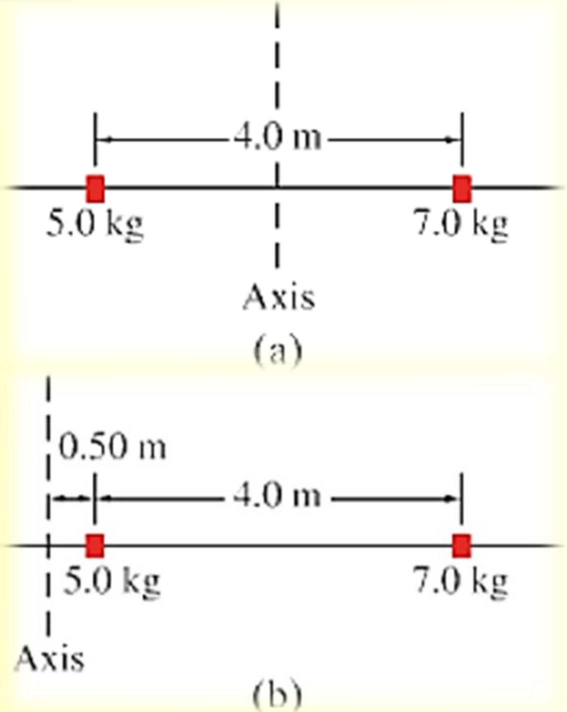
A disk is initially rotating counterclockwise around a fixed axis with angular speed ω_0. At time $t = 0$, the two forces shown in the figure below are exerted on the disk. If counter-clockwise is positive, which of the following could show the angular velocity of the disk as a function of time?





【Example】 Calculate Moment of Inertia

Two small "weights", of mass 5.0 kg and 7.0 kg, are mounted 4.0 m apart on a light rod (whose mass can be ignored).



Calculate the moment of inertia of system

(a) when rotated about an axis halfway between the weights?

(b) when rotated about an axis 0.50 m to the left of the 5.0-kg mass?

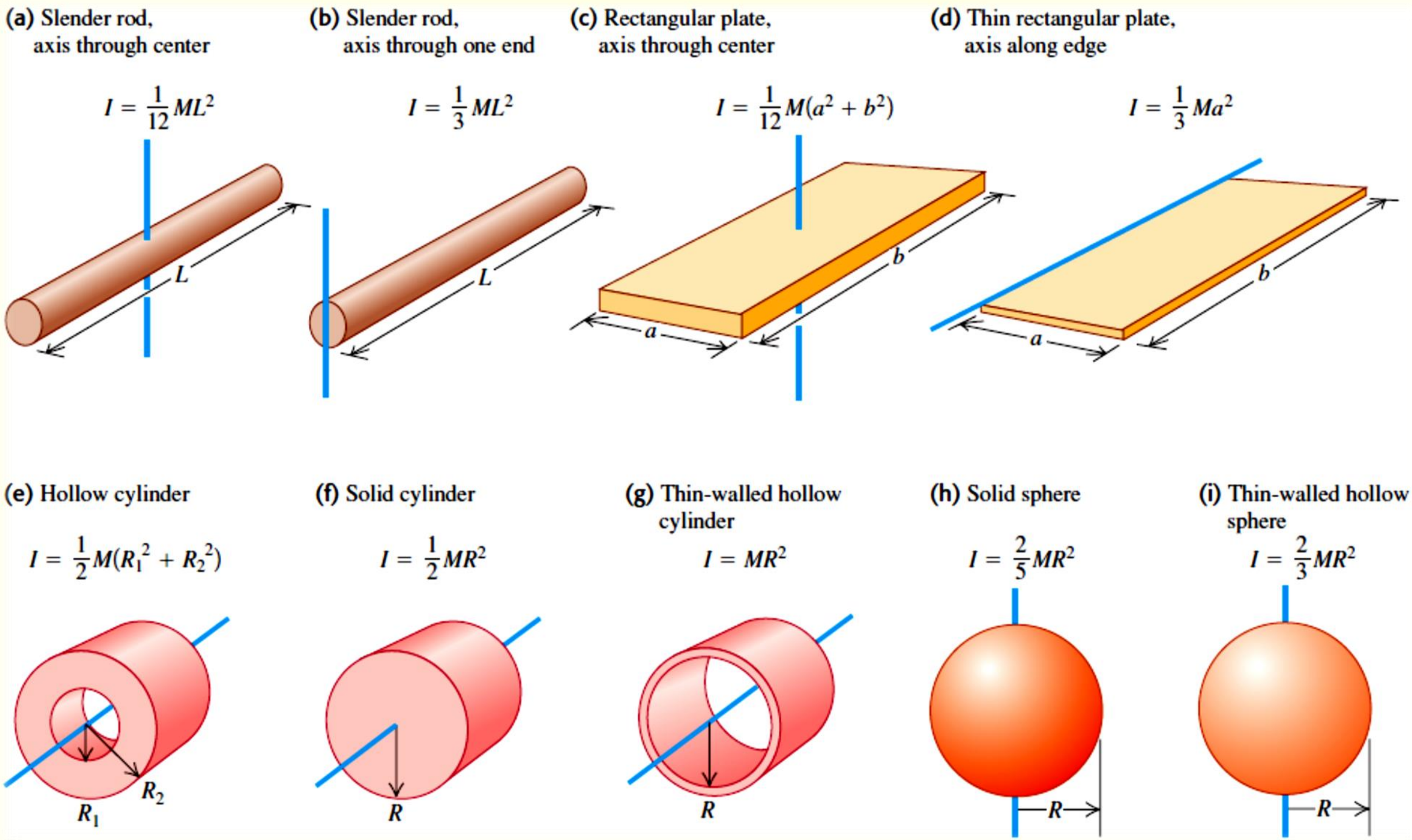
According to the definition of moment of inertia of the system,

(a) $I = \Sigma mr^2 = m_1 r_1^2 + m_2 r_2^2 = (5.0 \times 2.0^2 + 7.0 \times 2.0^2)\text{kg} \cdot \text{m}^2 = 48\text{kg} \cdot \text{m}^2$

(b) $I = \Sigma mr^2 = m_1 r_1^2 + m_2 r_2^2 = (5.0 \times 0.5^2 + 7.0 \times 4.5^2)\text{kg} \cdot \text{m}^2 = 143.3\text{kg} \cdot \text{m}^2$

***I* depends on axis of rotation, and on distribution of mass**

【Moments of inertia of various bodies】



The Two Conditions for Equilibrium

An object in mechanical equilibrium must satisfy the following two conditions:

Statement of translational equilibrium

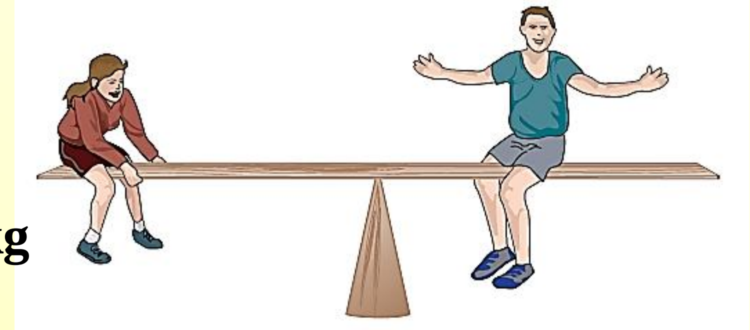
- ✓ **The net external force must be zero:** $\sum \vec{F} = 0$
- ✓ **The net external torque must be zero:** $\sum \vec{\tau} = 0$

Statement of rotational equilibrium

That is, for an object to be in equilibrium, it must both *translate* and *rotate* at *a constant rate*.

【Example】 Balancing Act

A woman of mass m = 55 kg sits on the left end of seesaw—a plank of length L = 4.00 m, pivoted in the middle as in Figure.

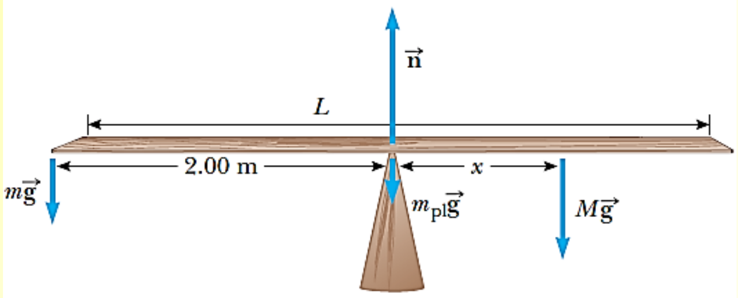


(a) First compute where a man of mass M = 75 kg sit if the system (seesaw + man + woman) is to be balanced?

(b) Find normal force exerted by the pivot if the plank has a mass of m_{pl} = 12 kg.

(c) Repeat (a), but compute the torques about an axis through left end of plank.

(a) Apply the rotational equilibrium to the plank by setting the net torque equal to zero:



$$\tau_{pivot} + \tau_{gravity} + \tau_{man} + \tau_{woman} = 0$$

$$\Downarrow$$

$$0 + 0 - Mgx + mg(L/2) = 0 \Rightarrow x = \frac{m(L/2)}{M} = 1.47\text{m}$$

(b) Apply the translational equilibrium to the plank (*set upward as positive*)**:**

$$-Mg - mg - m_{pl}g + n = 0 \Rightarrow n = (M + m + m_{pl})g = 1.42\times10^{3}\text{N}$$

(c) $\tau_{pivot} + \tau_{gravity} + \tau_{man} + \tau_{woman} = 0 \Rightarrow n(L/2) - m_{pl}g(L/2) - Mg(L/2 + x) + mg(0) = 0 \Rightarrow x = 1.46\text{m}$

Conservation of Angular Momentum

vector

L: angular momentum **SI unit: kg·m²/s**

Translation	Rotation
$p = mv$	$L = I\omega$
$\Sigma F = ma = \dfrac{\Delta p}{\Delta t}$	$\Sigma \tau = I\alpha = \dfrac{\Delta L}{\Delta t}$

Newton's Second Law for Rotation

When $\Sigma \tau = 0, \quad \dfrac{\Delta L}{\Delta t} = 0$

$$I\omega = I_0\omega_0 = const.$$

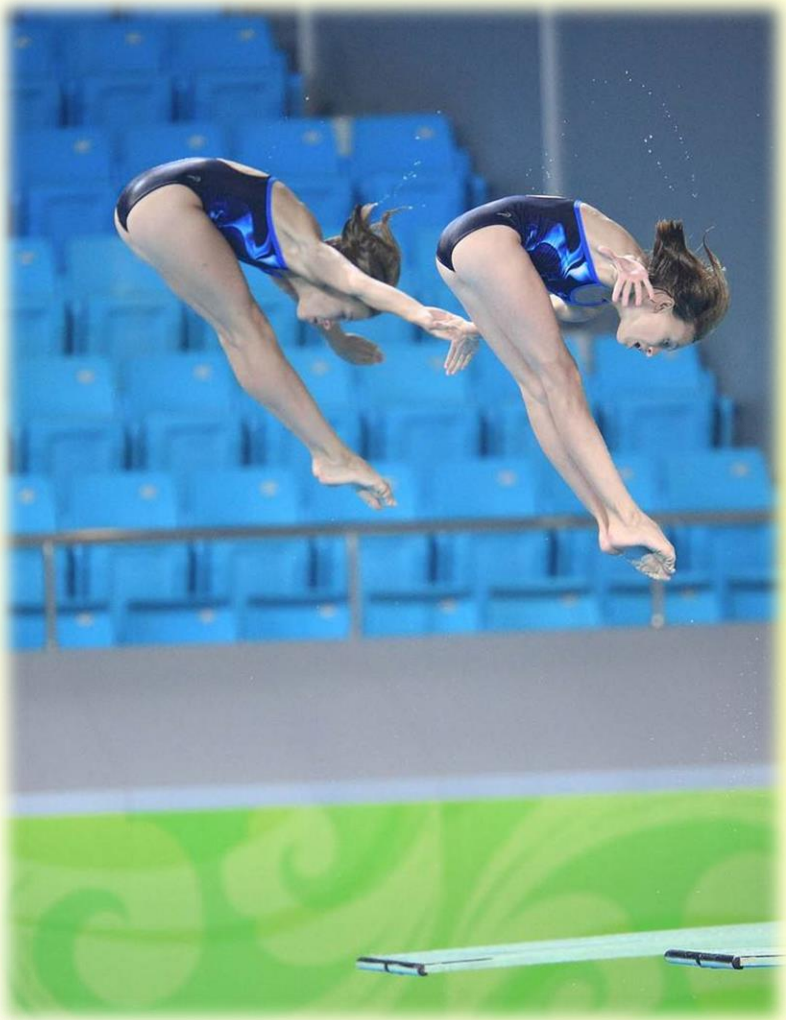
Conservation of Angular Momentum

(a) (b)

The figure skater controls her moment of inertia. (*a*) *By pulling in her arms and legs, she reduces her moment of inertia and increases her angular velocity* (*rate of spin*).
(*b*) *Upon landing, extending her arms and legs increases her moment of inertia and helps slow her spin*.

I small, ω large

I large, ω small



When a diver or an acrobat wishes to make several somersaults, she pulls her hands and feet close to the trunk of her body in order to rotate at a greater angular speed.

【Example】

A ball is tied to a string that passes through a glass tube and is attached to a hanging mass. The tube is held vertically as the ball is swung in a horizontal circle at a radius *r* and speed *v*.

If the string is pulled through the tube by hanging a larger mass on the string so that the radius becomes *r*/2, the speed of the ball will become

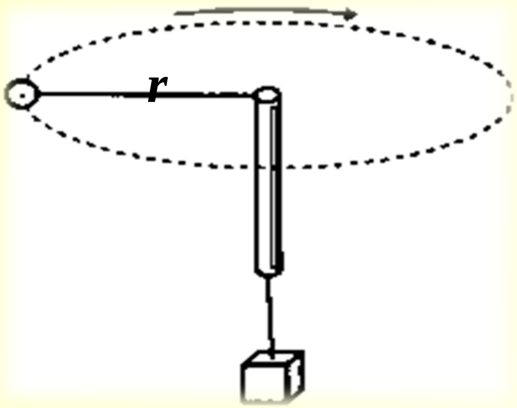
(A) *v*/2

(B) 2*v*

(C) 3*v*

(D) 4*v*

(E) *v*



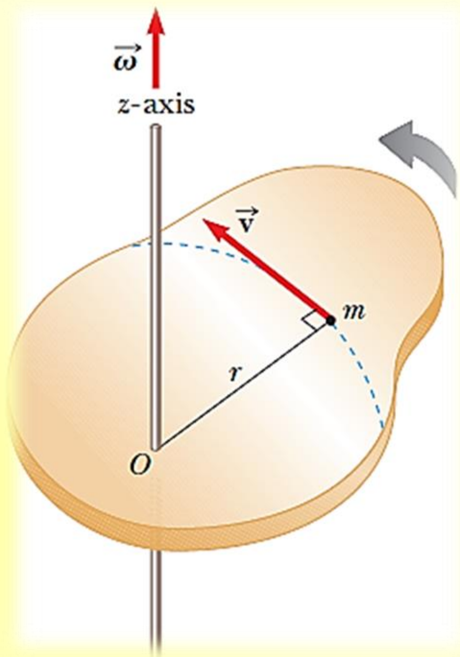
Rotational Kinetic Energy

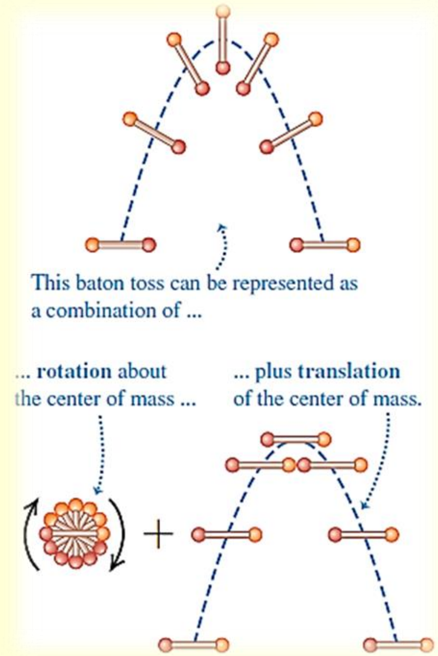
$$\left.\begin{aligned} K_{rot} &= \sum\left(\frac{1}{2}mv^2\right) \\ v &= r\omega \end{aligned}\right\}$$

$$K_{rot} = \sum\left(\frac{1}{2}mr^2\omega^2\right) = \frac{1}{2}\left(\sum mr^2\right)\omega^2$$

$= I$

$$K_{rot} = \frac{1}{2}I\omega^2$$





- Object with both translation and rotation:

$$K = K_{tra} + K_{rot} = \frac{1}{2}Mv_{\mathrm{cm}}^2 + \frac{1}{2}I_{\mathrm{cm}}\omega^2$$

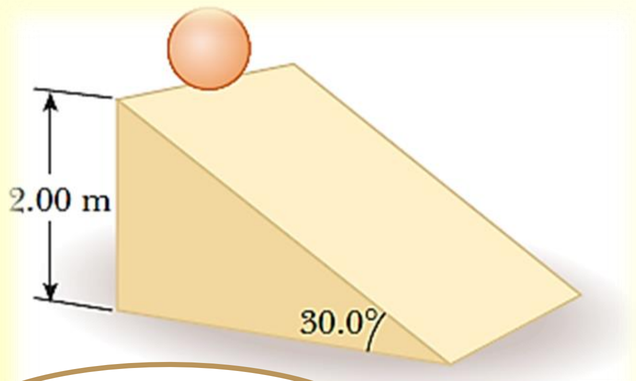
Conservation of mechanical energy of an isolated system:

$$(K_{tra} + K_{rot} + U)_i = (K_{tra} + K_{rot} + U)_f$$

【Example】 A Ball Rolling Down an Incline

A ball of mass M and radius R starts from rest at a height of 2.00 m and rolls down a 30.0° slope, as in Figure. What is the linear speed of the ball when it leaves the incline? Assume that the ball *rolls without slipping*? ($I_{cm} = \frac{2}{5}MR^2$)

Compare the result to that for an object *sliding* down a frictionless incline.



$$v = \sqrt{2gh} = 6.32\,\text{m/s}$$

Apply the conservation of mechanical energy to the ball,

$$(K_{tra} + K_{rot} + U)_i = (K_{tra} + K_{rot} + U)_f$$

$$0 + 0 + Mgh = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 + 0$$

$$I_{cm} = \frac{2}{5}MR^2$$

The ball rolls without slipping, so $v_{cm} = R\omega$

$$\Rightarrow v_{cm} = \sqrt{\frac{10}{7}gh} = 5.35\,\text{m/s}$$